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INDEX MATRICES AND OLAP-CUBE PART 1: APPLICATION OF THE INDEX MATRICES TO PRESENTATION OF OPERATIONS IN OLAP-CUBE

VELICHKA TRANEVA[1], VESSELINA BUREVA[1], EVDOKIA SOTIROVA[1]
KRASSIMIR ATANASSOV[1,2]
[1]"PROF. ASEN ZLATAROV" UNIVERSITY
"PROF. YAKIMOV" BLVD, BOURGAS 8000, BULGARIA
E-MAILS: *VELEKA13@GMAIL.COM*, *VBUREVA@GMAIL.COM*,
ENSOTIROVA@GMAIL.COM
[2] DEPARTMENT OF BIOINFORMATICS AND MATHEMATICAL MODELLING
INSTITUTE OF BIOPHYSICS AND BIOMEDICAL ENGINEERING
BULGARIAN ACADEMY OF SCIENCES
ACAD. G. BONCHEV STR., BL. 105, SOFIA-1113, BULGARIA
E-MAIL: *KRAT@BAS.BG*

Abstract. In this paper, we present an index matrix interpretation of the On-Line Analytal Processing (OLAP) cube. The aim is to present the basic OLAP operations with similar operations from the index matrix theory. The operations are discussed in terms of relational algebra and multidimensional models.

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1. Introduction

The Index Matrices (IM) were introduced in 1984 in [2] as an auxiliary tool for the description of the logic and functioning of generalized nets. Apparatus of 3D-Extended Index Matrices (3D-EIMs) was defined in a series of papers [3, 16, 17, 18] and book [4].

For the needs of the research we will present the definition of the 3D-EIMs and some operations over them in Section 2. In Section 3, will be presented definition of the OLAP-cube and its properties will be discussed. In this section, will be considered applications of the apparatus of the index matrices for presentation to basic operations in OLAP-cube.

2. Short remarks on 3D-extended index matrix

Let us start with the definition of the 3D-extended index matrix from [4, 8],which was extended in [16].

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2.1. **Definition of 3D-Extended index matrix and some operations over them.**

2.1.1. *Definition of 3D-Extended index matrix.* The Intuitionistic Fuzzy Pair (IFP) [5, 7] is an object in the form $\langle a, b\rangle$, where $a, b \in [0, 1]$ and $a+b \leq 1$, which is used as an evaluation of some object or process. Its components (a and b) are interpreted as degrees of membership and non-membership, or degrees of validity and non-validity, or degree of correctness and non-correctness, etc. Let I be a fixed set of indices,

$$I^n = \{\langle i_1, i_2, ..., i_n\rangle | (\forall j : 1 \leq j \leq n)(i_j \in I)\}$$

$$\text{and } I^* = \bigcup_{1\leq n\leq\infty} I^n.$$

Let X be a fixed set of some objects. In the particular cases, they can be either real numbers, or only the numbers 0 or 1, or logical variables, propositions or predicates, IFPs, function etc.

A "3D-extended Index Matrix" (3D-EIM) with index sets K, L and $H (K, L, H \subset I^*)$ and elements from set X is called the object:

$$[K, L, H, \{a_{k_i,l_j,h_g}\}] = \left\{ \begin{array}{c|ccccc} h_g & l_1 & \dots & l_j & \dots & l_n \\ \hline k_1 & a_{k_1,l_1,h_g} & \vdots & a_{k_1,l_j,h_g} & \dots & a_{k_1,l_n,h_g} \\ \vdots & \vdots & \dots & \vdots & \dots & \vdots \\ k_i & a_{k_i,l_1,h_g} & \dots & a_{k_i,l_j,h_g} & \dots & a_{k_i,l_n,h_g} \\ \vdots & \vdots & \dots & \vdots & \dots & \vdots \\ k_m & a_{k_m,l_1,h_g} & \dots & a_{k_m,l_j,h_g} & \dots & a_{k_m,l_n,h_g} \end{array} \quad | \; h_g \in H \right\}$$

where $K = \{k_1, k_2, \dots, k_m\}$, $L = \{l_1, l_2, \dots, l_n\}$, $H = \{h_1, h_2, \dots, h_f\}$, and for $1 \leq i \leq m,\ 1 \leq j \leq n,\ 1 \leq g \leq f : a_{k_i,l_j,h_g} \in X$.

Following [4, 16], let $3D - EIM_R$ be the set of all 3D-EIMs with elements being real numbers; $3D - EIM_{\{0,1\}}$ be the set of all 3D-EIMs with elements being 0 or 1; $3D - EIM_P$ be the set of all 3D-EIMs with elements – predicates; $3D - EIM_{IFP}$ be the set of all 3D-EIMs with elements – IFPs and $3D - EIM_{FE}$ – the set of all 3D-EIMs with elements – one-argument functions $\in F$ [1].

2.1.2. *Projection.* Let us have 3D-EIM $A = [K, L, H, \{a_{k_i,l_j,h_g}\}]$ and let $M \subseteq K$, $N \subseteq L$ and $U \subseteq H$. Then,

$$pr_{M,N,U}A = [M, N, U, \{b_{k_i,l_j,h_g}\}],$$

where for each $k_i \in M$, $l_j \in N$ and $h_g \in U$, $b_{k_i,l_j,h_g} = a_{k_i,l_j,h_g}$.

2.1.3. *Aggregation operations.* Let the 3D-EIM A $= [K, L, H, \{a_{k_i,l_j,h_g}\}]$ $(K, L, H \subset I^*)$ be given and let for $1 \leq i \leq m, 1 \leq j \leq n, 1 \leq g \leq f$, $\{k_{i,0}, k_0\} \notin K, \{l_{j,0}, l_0\} \notin L, \{h_{g,0}, h_0\} \notin H$ and $a_{k_i,l_j,h_g} \in X$. Let $\circ : X \times X \longrightarrow X$ and $* : X \times X \longrightarrow X$.

Let

$$\circ \in \begin{cases} \{``+", ``\times", ``average", ``max", ``min"\}, & \text{if } A \in 3D-EIM_R \\ & \text{or } A \in 3D-EIM_{FE}; \\ \{``max", ``min"\}, & \text{if } A \in 3D-EIM_{\{0,1\}} \\ \{``\wedge", ``\vee"\}, & \text{if } A \in 3D-EIM_P \\ & \text{or } A \in 3D-EIM_{IFP} \end{cases}.$$

In the case of $3D-EIM_{IFP}$, in aggregation operations participate aggregating pair operations $(\circ, *)$ whose elements are applied respectively to the first and second element of IFP, where

$$(\circ, *) \in \{(min, max), (min, average), (min, min), (average, average), (average, min), (max, min)\}.$$

Therefore, when $A \in 3D-EIM_{IFP}$, operations "$(\circ, *)$" are defined for the intuitionistic fuzzy pairs $\langle a, b\rangle$ and $\langle c, d\rangle$, elements of A by

$$\langle a, b\rangle(\circ, *)\langle c, d\rangle = \langle \circ(a, c), *(b, d)\rangle.$$

In all other cases, we use only one operation $(\circ)$.

The aggregation operations have the forms following [17]:

$(\circ)-\alpha_K$**-aggregation**

$$\alpha_{(K,\circ)}(A, k_0) = \left\{ \begin{array}{c|cccc} h_g & l_1 & l_2 & \dots & l_n \\ \hline k_0 & \underset{1\le i\le m}{\circ} a_{k_i,l_1,h_g} & \underset{1\le i\le m}{\circ} a_{k_i,l_2,h_g} & \dots & \underset{1\le i\le m}{\circ} a_{k_i,l_n,h_g} \end{array} | h_g \in H \right\}$$

$(\circ)-\alpha_L$**-aggregation**

$$\alpha_{(L,\circ)}(A, l_0) = \left\{ \begin{array}{c|c} h_g & l_0 \\ \hline k_1 & \underset{1\le j\le n}{\circ} a_{k_1,l_j,h_g} \\ k_2 & \underset{1\le j\le n}{\circ} a_{k_2,l_j,h_g} \\ \vdots & \vdots \\ k_m & \underset{1\le j\le n}{\circ} a_{k_m,l_j,h_g} \end{array} \mid h_g \in H \right\}$$

$(\circ)-\alpha_H$**-aggregation**

$$\alpha_{(H,\circ)}(A, h_0) = \left\{ \begin{array}{c|c} l_j & h_0 \\ \hline k_1 & \underset{1\le g\le f}{\circ} a_{k_1,l_1,h_g} \\ k_2 & \underset{1\le g\le f}{\circ} a_{k_2,l_2,h_g} \\ \vdots & \vdots \\ k_m & \underset{1\le g\le f}{\circ} a_{k_m,l_n,h_g} \end{array} \mid l_j \in L \right\}$$

$(\circ) - \alpha_{(K,L)}$**-aggregation**

$$\alpha_{(K,L,\circ)}(A, \langle k_0, l_0\rangle)$$

$$= \begin{array}{c|cccc} & h_1 & h_2 & \dots & h_f \\ \hline \langle k_0, l_0\rangle & \underset{\substack{1\le i\le m\\ 1\le j\le n}}{\circ} a_{k_i,l_j,h_1} & \underset{\substack{1\le i\le m\\ 1\le j\le n}}{\circ} a_{k_i,l_j,h_2} & \dots & \underset{\substack{1\le i\le m\\ 1\le j\le n}}{\circ} a_{k_i,l_j,h_f} \end{array};$$

$(\circ) - \alpha_{(K,H)}$**-aggregation**

$$\alpha_{(K,H,\circ)}(A, \langle k_0, h_0\rangle)$$

$$= \begin{array}{c|cccc} & l_1 & l_2 & \dots & l_n \\ \hline \langle k_0, h_0\rangle & \underset{\substack{1\le i\le m\\ 1\le g\le f}}{\circ} a_{k_i,l_1,h_g} & \underset{\substack{1\le i\le m\\ 1\le g\le f}}{\circ} a_{k_i,l_2,h_g} & \dots & \underset{\substack{1\le i\le m\\ 1\le g\le f}}{\circ} a_{k_i,l_n,h_g} \end{array};$$

$(\circ) - \alpha_{(L,H)}$**-aggregation**

$$\alpha_{(L,H,\circ)}(A, \langle l_0, h_0\rangle)$$

$$= \begin{array}{c|cccc} & k_1 & k_2 & \dots & k_m \\ \hline \langle l_0, h_0\rangle & \underset{\substack{1\le j\le n\\ 1\le g\le f}}{\circ} a_{k_1,l_j,h_g} & \underset{\substack{1\le j\le n\\ 1\le g\le f}}{\circ} a_{k_2,l_j,h_g} & \dots & \underset{\substack{1\le j\le n\\ 1\le g\le f}}{\circ} a_{k_m,l_j,h_g} \end{array}.$$

2.1.4. *Generalized aggregation operations over 3D-EIMs.* Let us remind the generalized aggregation operations from [15] as follows:
Let 3D-EIM A be given, that

$$[K, L, H, \{a_{k_{i,d},l_{j,b},h_{g,c}}\}]$$

$$= \left\{ \begin{array}{c|ccccccc} H_g \in H & L_1 & \dots & l_{j,1} & \dots & l_{j,J} & \dots & L_n \\ \hline K_1 & a_{K_1,L_1,H_g} & \vdots & a_{K_1,l_{j,1},H_g} & \dots & a_{K_1,l_{j,J},H_g} & \dots & a_{K_1,L_n,H_g} \\ \vdots & \vdots & \dots & \vdots & \dots & \vdots & \dots & \\ k_{i,1} & a_{k_{i,1},L_1,H_g} & \dots & a_{k_{i,1},l_{j,1},H_g} & \dots & a_{k_{i,1},l_{j,J},H_g} & \dots & a_{k_{i,1},L_n,H_g} \\ \vdots & \vdots & \dots & \vdots & \dots & \vdots & \dots & \vdots \\ k_{i,I} & a_{k_{i,I},L_1,H_g} & \dots & a_{k_{i,I},l_{j,1},H_g} & \dots & a_{k_{i,I},l_{j,J},H_g} & \dots & a_{k_{i,I},L_n,H_g} \\ \vdots & \vdots & \dots & \vdots & \dots & \vdots & \dots & \vdots \\ K_m & a_{K_m,L_1,H_g} & \dots & a_{K_m,l_{j,1},H_g} & \dots & a_{K_m,l_{j,J},H_g} & \dots & a_{K_m,L_n,H_g} \end{array} \right\}$$

$$\text{where } K = \{K_1, K_2, \dots, K_i, \dots, K_m\}, K_i = \{k_{i,1}, k_{i,2}, \dots, k_{i,I}\} \text{ for } 1 \le i \le m,$$
$$L = \{L_1, L_2, \dots, L_j, \dots, L_n\}, L_j = \{l_{j,1}, l_{j,2}, \dots, l_{j,J}\} \text{ for } 1 \le j \le n,$$
$$H = \{H_1, H_2, \dots, H_g, \dots, H_f\}, H_g = \{h_{g,1}, h_{g,2}, \dots, h_{g,G}\} \text{ for } 1 \le g \le f$$

and $(\{K, L, H\} \subset I^*)$, and for $1 \le i \le m,\ 1 \le j \le n,\ 1 \le g \le f,\ 1 \le d \le I,$
$1 \le b \le J,\ 1 \le c \le G : a_{k_{i,d},l_{j,b},h_{g,c}} \in X$.

The generalized aggregation operations over the given matrix A have the forms:
$(\circ) - \alpha_{(K,K_i)}$**-aggregation** – it is aggregation of index K_i of dimension K

$$\alpha_{(K,K_i,\circ)}(A, K_{i,0})$$

$$= \left\{ \begin{array}{c|ccccc} H_g \in H & L_1 & \dots & L_j & \dots & L_n \\ \hline K_1 & a_{K_1,L_1,H_g} & \dots & a_{K_1,L_j,H_g} & \dots & a_{K_1,L_n,H_g} \\ \vdots & \vdots & \dots & \vdots & \dots & \vdots \\ K_{i,0} & \underset{i,1\le i\le i,I}{\circ} a_{k_i,L_1,H_g} & \dots & \underset{i,1\le i\le i,I}{\circ} a_{k_i,L_j,H_g} & \dots & \underset{i,1\le i\le i,I}{\circ} a_{k_i,L_n,H_g} \\ \vdots & \vdots & \dots & \vdots & \dots & \vdots \\ K_m & a_{K_m,L_1,H_g} & \dots & a_{K_m,L_j,H_g} & \dots & a_{K_m,L_n,H_g} \end{array} \right\}$$

where $K_i \subset K$ and $1 \le i \le m$

Let index set $K_* \subseteq K$ be given and $K_* = \{K_{v_1}, \dots, K_{v_x}, \dots, K_{v_t}\}, 1 \le v_x \le m$ for $1 \le x \le t$; $V_* = \{K_{v_1,0}, \dots, K_{v_x,0}, \dots, K_{v_t,0}\}, K_{v_x,0} \notin K$ for $1 \le x \le t$.
Let us recall the following definitions: $(\circ) - \alpha_{(K,K_*)}$**-aggregation**

$$\alpha_{(K,K_*,\circ)}(A, V_*) = \alpha_{(K,K_*,\circ)}(A, \langle K_{v_{1,0}}, \dots, K_{v_{x,0}}, \dots, K_{v_{p,0}}\rangle)$$
$$= \alpha_{(K,K_{V_t},\circ)}((\dots \alpha_{(K,K_{v_1},\circ)}(A, K_{v_{1,0}}) \dots), K_{v_{t,0}}).$$

Analogically are constructed the definitions of the operations:

$\{(\circ) - \alpha_{(L,L_j)}\}, \{(\circ) - \alpha_{(H,H_g)}\}$-aggregation and their summaries.

Let us present the definitions for other aggregating operations from [15]:

$(\circ) - \alpha_{(\langle K,K_i\rangle,\langle L,L_j\rangle)}$**-aggregation**

$$\alpha_{(\langle K,K_i\rangle,\langle L,L_j\rangle,\circ)}(A, \langle K_{i,0}, L_{j,0}\rangle)$$

$$= \left\{ \begin{array}{c|ccccc} H_g \in H & L_1 & \dots & L_{j,0} & \dots & L_n \\ \hline K_1 & a_{K_1,L_1,H_g} & \dots & \underset{j,1\le j\le j,J}{\circ} a_{K_1,l_j,H_g} & \dots & a_{K_1,L_n,H_g} \\ \vdots & \vdots & \dots & \vdots & \dots & \vdots \\ K_{i,0} & \underset{i,1\le i\le i,I}{\circ} a_{k_i,L_1,H_g} & \dots & \underset{\substack{i,1 \le i \le i,I \\ j,1 \le j \le j,J}}{\circ} a_{k_i,l_j,H_g} & \dots & \underset{i,1\le i\le i,I}{\circ} a_{k_i,L_n,H_g} \\ \vdots & \vdots & \dots & \vdots & \dots & \vdots \\ K_m & a_{K_m,L_1,H_g} & \dots & a_{K_m,L_j,H_g} & \dots & a_{K_m,L_n,H_g} \end{array} \right\}$$

where $K_i \subset K$ and $L_j \subset L$.

Let index sets $K_* \subseteq K$ be given and $K_* = \{K_{v_1}, \dots, K_{v_x}, \dots, K_{v_t}\}, 1 \le v_x \le m$ for $1 \le x \le t$ and $L_* \subseteq L$ and $L_* = \{L_{u_1}, \dots, L_{u_y}, \dots, L_{u_s}\}, 1 \le u_y \le n$ for $1 \le y \le s$. Let be given $V_* = \{K_{v_1,0}, \dots, K_{v_x,0}, \dots, K_{v_t,0}\}, K_{v_x,0} \notin K$ for $1 \le x \le t$; $W_* = \{L_{u_{1,0}}, \dots, L_{u_{y,0}}, \dots, L_{u_{s,0}}\}, L_{u_y,0} \notin L$ for $1 \le y \le s$.

Let us recall the definition of:

$(\circ) - \alpha_{(\langle K,K_*\rangle,\langle L,L_*\rangle)}$**-aggregation**

$$\alpha_{(\langle K,K_*\rangle,\langle L,L_*\rangle,\circ)}(A,V_*,W_*)$$
$$= \alpha_{(\langle K,K_*\rangle,\langle L,L_*\rangle,\circ)}(A,\langle K_{v_{1,0}},\dots,K_{v_{x,0}},\dots,K_{v_{t,0}}\rangle,\langle L_{u_{1,0}},\dots,L_{u_{y,0}},\dots,L_{u_{s,0}}\rangle)$$
$$= \alpha_{(\langle K,K_{v_p}\rangle,\langle L,L_{u_s}\rangle,\circ)}((\dots\alpha_{(\langle K,K_{v_1}\rangle,\langle L,L_{u_1}\rangle,\circ)}(A,\langle K_{v_{1,0}},L_{u_{1,0}}\rangle)\dots),\langle K_{v_{t,0}},L_{u_{s,0}}\rangle).$$

Analogically are constructed the definitions of following aggregation operations: $\{(\circ) - \alpha_{(\langle K,K_i\rangle,\langle H,H_g\rangle)}\}$, $\{(\circ) - \alpha_{(\langle L,L_j\rangle,\langle H,H_g\rangle)}\}$-aggregation and their summaries.

2.1.5. *Hierarchical operators over EIMs.* In [4, 6], three hierarchical operators are defined by Atanassov over 2D-EIMs, when their elements are not only numbers, variables, etc, but when they also can be whole (separate) IMs. Let us generalize these three operators over 3D-EIMs. Let $A = [K, L, H, \{a_{k_i,l_j,h_g}\}$ be an 3D-EIM and let its element a_{k_x,l_y,h_z} be an IM by itself:

$$a_{k_x,l_y,h_z} = [P, Q, O\{b_{p_r,q_s,o_e}\}],$$

where

$$K \cap P = L \cap Q = H \cap O = \emptyset.$$

Here, we will extend the definition of the first hierarchical operator:

$$A|(a_{k_x,l_y,h_z}) = [(K - \{k_x\}) \cup P, (L - \{l_y\}) \cup Q, (H - \{h_z\}) \cup O, \{c_{t_u,v_w,s_d}\}],$$

where

$$c_{t_u,v_w,s_d} = \begin{cases} a_{k_i,l_j,h_g}, & \text{if } t_u = k_i \in K - \{k_x\}, v_w = l_j \in L - \{l_y\} \text{ and } s_d = h_g \in H - \{h_z\} \\ b_{p_r,q_s,o_e}, & \text{if } t_u = p_r \in P, v_w = q_s \in Q \text{ and } s_d = o_e \in O \\ 0, & \text{otherwise} \end{cases}.$$

Let for $i = 1, 2, ..., m$:

$$a^i_{k_{i,x},l_{i,y},h_{i,z}} = [P_i, Q_i, O_i, \{b^i_{p_{i,r},q_{i,s},o_{i,e}}\}],$$

where for every i, j $(1 \le i < j \le m)$:

$$P_i \cap P_j = Q_i \cap Q_j = O_i \cap O_j = \emptyset,$$

$$P_i \cap K = Q_i \cap L = O_i \cap H = \emptyset.$$

Then, for $k_{1,x}, k_{2,x}, ..., k_{m,x} \in K$, $l_{1,y}, l_{2,y}, ..., l_{m,y} \in L$ and $h_{1,z}, h_{2,z}, ..., h_{m,z} \in H$:

$$A|(a^1_{k_{1,x},l_{1,y},h_{1,z}}, a^2_{k_{2,x},l_{2,y},h_{2,z}}, ..., a^m_{k_{m,x},l_{m,y},h_{m,z}})$$
$$= (...((A|(a^1_{k_{1,x},l_{1,y},h_{1,z}}))|(a^2_{k_{2,x},l_{2,y},h_{2,z}}))...)|(a^m_{k_{m,x},l_{m,y},h_{m,z}})).$$

The second form of this operator for the above defined IM A and its fixed element a_{k_x,l_y,h_z}, is

$$A|^*(a_{k_x,l_y,h_z})$$

$$
= \begin{array}{l|cccc c}
h_g \in H & l_1 & \dots & l_{y-1} & q_1 & \dots \\
\hline
k_1 & a_{k_1,l_1,h_g} & \dots & a_{k_1,l_{y-1},h_g} & a_{k_1,l_y,h_g} & \dots \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots \\
k_{x-1} & a_{k_{x-1},l_1,h_g} & \dots & a_{k_{x-1},l_{y-1},h_g} & a_{k_{x-1},l_y,h_g} & \dots \\
p_1 & a_{k_x,l_1,h_g} & \dots & a_{k_x,l_{y-1},h_g} & b_{p_1,q_1,o_e} & \dots \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots \\
p_u & a_{k_x,l_1,h_g} & \dots & a_{k_x,l_{y-1},h_g} & b_{p_u,q_1,o_e} & \dots \\
k_{x+1} & a_{k_{x+1},l_1,h_g} & \dots & a_{k_{x+1},l_{y-1},h_g} & a_{k_{x+1},l_y,h_g} & \dots \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots \\
k_m & a_{k_m,l_1,h_g} & \dots & a_{k_m,l_{y-1},h_g} & a_{k_m,l_y,h_g} & \dots
\end{array}
$$

$$
\begin{array}{ccccc}
\dots & q_v & l_{y+1} & \dots & l_n \\
\hline
\dots & a_{k_1,l_y,h_g} & a_{k_1,l_{y+1},h_g} & \dots & a_{k_1,l_n,h_g} \\
\ddots & \vdots & \vdots & \vdots & \vdots \\
\dots & a_{k_{x-1},l_y,h_g} & a_{k_{x-1},l_{y+1},h_g} & \dots & a_{k_{x-1},l_n,h_g} \\
\dots & b_{p_1,q_v,o_e} & a_{k_x,l_{y+1},h_g} & \dots & a_{k_x,l_n,h_g} \\
\ddots & \vdots & \vdots & \vdots & \vdots \\
\dots & b_{p_u,q_v,o_e} & a_{k_x,l_{y+1},h_g} & \dots & a_{k_x,l_n,h_g} \\
\dots & a_{k_{x+1},l_y,h_g} & a_{k_{x+1},l_{y+1},h_g} & \dots & a_{k_{x+1},l_n,h_g} \\
\ddots & \vdots & \vdots & \vdots & \vdots \\
\dots & a_{k_m,l_y,h_g} & a_{k_m,l_{y+1},h_g} & \dots & a_{k_m,l_n,h_g}
\end{array}.
$$

Let us present the third hierarchical operator for the above IM $A = [K, L, H, \{a_{k_i,l_j,h_g}\}]$, where $K = \{k_1, k_2, \dots, k_m\}, L = \{l_1, l_2, \dots, l_n\}, H = \{h_1, h_2, \dots, h_f\}$ and for $1 \le i \le m, 1 \le j \le n$ and $1 \le g \le f : a_{k_i,l_j,h_g} \in X$.

Let each element a_{k_i,l_j,h_g} be an IM by itself:
$a_{k_i,l_j,h_g} = [P_{k_i,l_j,h_g}, Q_{k_i,l_j,h_g}, W_{k_i,l_j,h_g}, \{b_{k_i,l_j,h_g,p_u,q_v,w_\pi}\}]$, where $1 \le i \le m, 1 \le j \le n, 1 \le g \le f, 1 \le u \le U_{i,j,g}, 1 \le v \le V_{i,j,g}, 1 \le \pi \le \Pi_{i,j,g}$ and $P_{k_i,l_j,h_g} = \{p_{k_i,l_j,h_g,1}, \dots, p_{k_i,l_j,h_g,U_{i,j,g}}\}, Q_{k_i,l_j,h_g} = \{q_{k_i,l_j,h_g,1}, \dots, q_{k_i,l_j,h_g,V_{i,j,g}}\}$, $W_{k_i,l_j,h_g} = \{w_{k_i,l_j,h_g,1}, \dots, w_{k_i,l_j,h_g,\Pi_{i,j,g}}\}$, $K \cap P_{k_i,l_j,h_g} = L \cap Q_{k_i,l_j,h_g} = H \cap W_{k_i,l_j,h_g} = \emptyset$ and for every six indices $\{k_i, k_x\} \in K$, $\{l_j, l_y\} \in L$ and $\{h_g, h_z\} \in H : P_{k_i,l_j,h_g} \times Q_{k_i,l_j,h_g} \cap P_{k_x,l_y,h_z} \times Q_{k_x,l_y,h_z} = \emptyset$, i.e., there is no pair of indices that is found in two different IMs in the given EIM A, where "$\times$" is the standard Cartesian product. In this case, if there are two or more $p-$indices in $A|^*$ that coincide, all members of the first dimension with these indices are written on the respective places in the first dimension with coinciding indices. In this case, if there are two or more $q-$indices in $A|^*$ that coincide, all members of the second dimension with these indices are written on the respective places in the second column with coinciding indices. In this case, if there are two or more $w-$indices in $A|^*$ that coincide, all members of the third dimension with these indices are written on the respective places in the third column with coinciding indices.

$$A|^{*}$$

$$= \begin{array}{c|ccccc}
h_g \in H & q_{k_1,l_1,h_g,1} & \dots & q_{k_1,l_1,h_g,V_{1,1,g}} & \dots \\ \hline
p_{k_1,l_1,h_g,1} & a_{k_1,l_1,h_g,1,1,w_{k_i,l_j,h_g,\pi}} & \dots & a_{k_1,l_1,h_g,1,V_{1,1,g},w_{k_i,l_j,h_g,\pi}} & \dots \\
\vdots & \vdots & \ddots & \vdots & \ddots \\
p_{k_1,l_1,h_g,U_{1,1,g}} & a_{k_1,l_1,h_g,U_{1,1,g},1,w_{k_i,l_j,h_g,\pi}} & \dots & a_{k_1,l_1,h_g,U_{1,1,g},V_{1,1,g},w_{k_i,l_j,h_g,\pi}} & \dots \\
\vdots & \vdots & \ddots & \vdots & \ddots \\
p_{k_m,l_n,h_g,1} & a_{k_m,l_1,h_g,1,1,w_{k_i,l_j,h_g,\pi}} & \dots & a_{k_m,l_1,h_g,1,V_{1,1,g},w_{k_i,l_j,h_g,\pi}} & \dots \\
\vdots & \vdots & \ddots & \vdots & \ddots \\
p_{k_m,l_n,h_g,U_{m,n,g}} & a_{k_m,l_n,h_g,U_{1,1,g},1,w_{k_i,l_j,h_g,\pi}} & \dots & a_{k_m,l_n,h_g,U_{1,1,g},V_{1,1,g},w_{k_i,l_j,h_g,\pi}} & \dots
\end{array}$$

$$\begin{array}{ccc}
q_{k_m,l_n,h_g,1} & \dots & q_{k_m,l_n,h_g,V_{m,n,g}} \\ \hline
a_{k_1,l_n,h_g,1,1,w_{k_i,l_j,h_g,\pi}} & \dots & a_{k_1,l_n,h_g,1,V_{m,n,g},w_{k_i,l_j,h_g,\pi}} \\
\vdots & \ddots & \vdots \\
a_{k_1,l_n,h_g,U_{1,1,g},1,w_{k_i,l_j,h_g,\pi}} & \dots & a_{k_1,l_n,h_g,U_{1,n,g},V_{m,n,g},w_{k_i,l_j,h_g,\pi}} \\
\vdots & \ddots & \vdots \\
a_{k_m,l_n,h_g,1,1,w_{k_i,l_j,h_g,\pi}} & \dots & a_{k_m,l_n,h_g,1,U_{m,n,g},w_{k_i,l_j,h_g,\pi}} \\
\vdots & \ddots & \vdots \\
a_{k_m,l_n,h_g,U_{m,n,g},1,w_{k_i,l_j,h_g,\pi}} & \dots & a_{k_m,l_n,h_g,U_{m,n,g},V_{m,n,g},w_{k_i,l_j,h_g,\pi}}
\end{array},$$

where $w_{k_i,l_j,h_g,\pi} \in W_{k_i,l_j,h_g}$. The matrix is padding ashing with zeros if an index set by one dimension of the submatrix is shorter / longer than that of another submatrix.

2.2. **Definition of 3D-Multilayer extended index matrix and some operations over them** [15].

2.2.1. *Definition of 3D-multilayer extended index matrix (3D-MLEIM).* Let us begin this part with a definition of a 3D-multilayer extended index matrix A (3D-MLEIM) with P levels (layers) of use of a dimension K, Q levels(layers) of use of a dimension L and R levels(layers) of use of a dimension H as follows:

$$A = [K, L, H, \{a_{K_{i,d}^{(p)}, L_{j,b}^{(q)}, H_{g,c}^{(r)}}\}]$$

$$= \left\{ \begin{array}{c|ccccc}
H_g^{(R)} \in H & L_1^{(Q)} & \dots & L_j^{(Q)} & \dots & L_n^{(Q)} \\ \hline
K_1^{(P)} & a_{K_1^{(P)},L_1^{(Q)},H_g^{(R)}} & \dots & a_{K_1^{(P)},L_j^{(Q)},H_g^{(R)}} & \dots & a_{K_1^{(P)},L_n^{(Q)},H_g^{(R)}} \\
\vdots & \vdots & \dots & \vdots & \dots & \vdots \\
K_i^{(P)} & a_{K_i^{(P)},L_1^{(Q)},H_g^{(R)}} & \dots & a_{K_i^{(P)},L_j^{(Q)},H_g^{(R)}} & \dots & a_{K_i^{(P)},L_n^{(Q)},H_g^{(R)}} \\
\vdots & \vdots & \dots & \vdots & \dots & \vdots \\
K_m^{(P)} & a_{K_m^{(P)},L_1^{(Q)},H_g^{(R)}} & \dots & a_{K_m^{(P)},L_j^{(Q)},H_g^{(R)}} & \dots & a_{K_m^{(P)},L_n^{(Q)},H_g^{(R)}}
\end{array} \right\}$$

where

$$K = \{K_1^{(P)}, K_2^{(P)}, \dots, K_i^{(P)}, \dots, K_m^{(P)}\},$$

$$K_i^{(P)} = \left\{K_{i,1}^{(P-1)}, K_{i,2}^{(P-1)}, \dots, K_{i,x}^{(P-1)}, \dots, K_{i,I}^{(P-1)}\right\} \text{ for } 1 \le i \le m$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$K_u^{(1)} = \left\{K_{u,1}^{(0)}, K_{u,2}^{(0)}, \dots, K_{u,U}^{(0)}\right\}$$

i.e. p-th layer of dimension K of the multilayer matrix, where $(1 \le p \le P)$, is presented by

$$K_{u_*}^{(p)} = \left\{K_{u_*,1}^{(p-1)}, K_{u_*,2}^{(p-1)}, \dots, K_{u_*,U_*}^{(p-1)}\right\} \text{ for } 1 \le p \le P$$

$$L = \{L_1^{(Q)}, L_2^{(Q)}, \dots, L_j^{(Q)}, \dots, L_n^{(Q)}\},$$

$$L_j^{(Q)} = \{L_{j,1}^{(Q-1)}, L_{j,2}^{(Q-1)}, \dots, L_{j,y}^{(Q-1)}, \dots, L_{j,J}^{(Q-1)}\} \text{ for } 1 \le j \le n$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$L_v^{(1)} = \left\{L_{v,1}^{(0)}, L_{v,2}^{(0)}, \dots, l_{v,V}^{(0)}\right\}$$

i.e. q-th layer of dimension Q of the multilayer matrix is presented by

$$L_{v_*}^{(q)} = \left\{L_{v_*,1}^{(q-1)}, L_{v_*,2}^{(q-1)}, \dots, L_{v_*,V_*}^{(q-1)}\right\} \text{ for } 1 \le q \le Q$$

$$H = \{H_1^{(R)}, H_2^{(R)}, \dots, H_g^{(R)}, \dots, H_f^{(R)}\},$$

$$H_g^{(R)} = \{H_{g,1}^{(R-1)}, H_{g,2}^{(R-1)}, \dots, H_{g,z}^{(R-1)}, \dots, H_{g,G}^{(R-1)}\} \text{ for } 1 \le g \le f$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$H_w^{(1)} = \left\{H_{w,1}^{(0)}, H_{w,2}^{(0)}, \dots, H_{w,W}^{(0)}\right\}$$

i.e. r-th layer of dimension H of the multilayer matrix is presented by

$$H_{w_*}^{(r)} = \left\{H_{w_*,1}^{(r-1)}, H_{w_*,2}^{(r-1)}, \dots, H_{w_*,W_*}^{(r-1)}\right\} \text{ for } 1 \le r \le R$$

and $(K, L, H \subset I^*)$, and for $1 \le i \le I,\ 1 \le j \le J,\ 1 \le g \le G,\ 1 \le p \le P,$ $1 \le q \le Q,\ 1 \le r \le R,\ 1 \le d \le I,\ 1 \le b \le J,\ 1 \le c \le G : a_{K_{i,d}^{(p)}, L_{j,b}^{(q)}, H_{g,c}^{(r)}} \in X,$ $K_{i,0}^{(p)} \notin K,\ L_{j,0}^{(q)} \notin L$ and $H_{g,0}^{(r)} \notin H.$

2.2.2. *Generalized aggregation operations over 3D-multilayer extended index matrix (3D-MLEIM).* The definition of the generalized aggregation operation, which presents aggregation on the p-th layer of dimension K of the matrix A, which is 3D-MLEIM, is [15]:

$(\circ) - \alpha_{(K,K_i^{(P)},p\ \textbf{-layer})}$**-aggregation**

$$\alpha_{(K,K_i^{(P)},p\ \text{-layer}\ ,\circ)}(A, K_{i,0}^{(p)})$$

$$
= \left\{ \begin{array}{c|ccccccc}
H_g^{(R)} \in H & L_1^{(Q)} & \dots & \dots & L_j^{(Q)} & \dots & L_n^{(Q)} \\ \hline
K_1^{(P)} & a_{K_1^{(P)}, L_1^{(Q)}, H_g^{(R)}} & \dots & \dots & a_{K_1^{(P)}, L_j^{(Q)}, H_g^{(R)}} & \dots & a_{K_1^{(P)}, L_n^{(Q)}, H_g^{(R)}} \\
\vdots & \vdots & \ddots & \ddots & \vdots & \ddots & \vdots \\
K_i^P & \dots & \ddots & & & & \\
\vdots & \vdots & \ddots & & & & \\
\{K_i^P, \text{ p-layer }\} K_{i,0}^{(p)} & \underset{\substack{1 \le \rho \le p-1 \\ K_u^{(\rho)} \in K_{i*}^{(p)}}}{\circ} a_{K_u^{(\rho)}, L_1^{(Q)}, H_g^{(R)}} & \dots & \dots & \underset{\substack{1 \le \rho \le p-1 \\ K_u^{(\rho)} \in K_{i*}^{(p)}}}{\circ} a_{K_u^{(\rho)}, L_j^{(Q)}, H_g^{(R)}} & \dots & \underset{\substack{1 \le \rho \le p-1 \\ K_u^{(\rho)} \in K_{i*}^{(p)}}}{\circ} a_{K_u^{(\rho)}, L_n^{(Q)}, H_g^{(R)}} \\
\vdots & \vdots & \ddots & \ddots & \vdots & \ddots & \vdots \\
K_m^{(P)} & a_{K_m^{(P)}, L_1^{(Q)}, H_g^{(R)}} & \dots & \dots & a_{K_m{}^{(P)}, L_j^{(Q)}, H_g^{(R)}} & \dots & a_{K_m^{(P)}, L_n^{(Q)}, H_g^{(R)}}
\end{array} \right\}
$$

where $K_i^{(P)} \subset K, 1 \le p \le P$

Let index set $K_* \subseteq K$ be given and $K_* = \{K_{v_1}^{(P)}, \dots, K_{v_x}^{(P)}, \dots, K_{v_t}^{(P)}\}$,

$V_* = \{K_{v_1,0}^{(p)}, \dots, K_{v_x,0}^{(p)}, \dots, K_{v_t,0}^{(p)}\} \notin K$ for $1 \le p \le P$.

In this case let us recall the definition of the aggregation operation:

$(\circ) - \alpha_{(K, K_*, p \text{ -layer})}$**-aggregation**

$\alpha_{(K, K_*, p \text{ -layer}, \circ)}(A, V_*) = \alpha_{(K, K_*, \text{ p-layer}, \circ)}(A, \langle K_{v_1,0}^{(p)}, \dots, K_{v_x,0}^{(p)}, \dots, K_{v_t,0}^{(p)} \rangle)$ $= \alpha_{(K, K_{v_t}^{(P)}, \circ)}((\dots \alpha_{(K, K_{v_1}^{(P)}, \circ)}(A, K_{v_1,0}^{(p)}) \dots), K_{v_t,0}^{(p)})$. Let index set $K_* \subseteq K$ be given and $K_* = \{K_{v_1}^{(P)}, \dots, K_{v_x}^{(P)}, \dots, K_{v_t}^{(P)}\}$, $V_* = \{K_{v_1,0}^{(p_1)}, \dots, K_{v_x,0}^{(p_x)}, \dots, K_{v_t,0}^{(p_t)}\} \notin K$ and $P_* = \{p_1, \dots, p_x, \dots, p_t\}$, where $1 \le p_x \le P$ for $1 \le x \le t$. We denote the power set (the number of elements of the set G) of G by $|G| = u$. Let $|K_*| = |P_*| = |V_*| = t$.

In this case we will recall the definition of aggregation operation as follows:

$(\circ) - \alpha_{(K, K_*, P_*)}$**-aggregation**

$\alpha_{(K, K_*, P_*, \circ)}(A, V_*) = \alpha_{(K, K_*, P_*, \circ)}(A, \langle K_{v_1,0}^{(p_1)}, \dots, K_{v_x,0}^{(p_x)}, \dots, K_{v_t,0}^{(p_t)} \rangle)$
$= \alpha_{(K, K_{v_t}^{(P)}, \circ)}((\dots \alpha_{(K, K_{v_1}^{(P)}, \circ)}(A, K_{v_1,0}^{(p_1)}) \dots), K_{v_t,0}^{(p_t)})$.

Similar are the definitions of following operations: $(\circ) - \alpha_{(L, L_j^{(Q)}, q \text{ -layer})}$, $(\circ) - \alpha_{(H, H_g^{(R)}, r \text{ -layer})}$-aggregation and their generalizations.

Let us remind the definitions for other aggregating operations from [15]:
$\alpha_{(\langle K,K_i^{(P)},p\text{ -layer }\rangle,\langle L,L_j^{(Q)},q\text{ -layer }\rangle,\circ)}$, $\alpha_{(\langle K,K_i^{(P)},p\text{ -layer }\rangle,\langle H,H_g^{(R)},r\text{ -layer }\rangle,\circ)}$ and $\alpha_{(\langle L,L_j^{(Q)},q\text{ -layer }\rangle,\langle H,H_g^{(R)},r\text{ -layer }\rangle,\circ)}$-aggregation.

The definition of $\{(\circ)-\alpha_{(\langle K,K_i^{(P)},p\ \textbf{-layer}\ \rangle,\langle L,L_j^{(Q)},q\ \textbf{-layer}\ \rangle,\circ)}$**-aggregation**$\}$ is:

$$\alpha_{(\langle K,K_i^{(P)},p\text{ -layer }\rangle,\langle L,L_j^{(Q)},q\text{ -layer }\rangle,\circ)}(A,\langle K_{i,0}^{(p)},L_{j,0}^{(q)}\rangle)=$$

$$\left\{\begin{array}{c|ccccc}
 & L_1^{(Q)} & \dots & L_{j,0}^{(q)} & \dots & L_n^{(Q)} \\ \hline
K_1^{(P)} & a_{K_1^{(P)},L_1^{(Q)},H_g^{(R)}} & \dots & \underset{\substack{1\le\sigma\le q-1\\ L_v^{\sigma}\in L_{j*}^{(q)}}}{\circ} a_{K_1^{(P)},L_j^{(\sigma)},H_g^{(R)}} & \dots & a_{K_1^{(P)},L_n^{(Q)},H_g^{(R)}} \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
K_{i,0}^{(p)} & \underset{\substack{1\le\rho\le p-1\\ K_u^{\rho}\in K_{i*}^{(p)}}}{\circ} a_{K_u^{(\rho)},L_1^{(Q)},H_g^{(R)}} & \dots & \underset{\substack{1\le\rho\le P-1\\ K_u^{\rho}\in K_{i*}^{(p)}\\ 1\le\sigma\le Q-1\\ L_v^{\sigma}\in L_{j*}^{(q)}}}{\circ} a_{K_u^{(\rho)},L_j^{(\sigma)},H_g^{(R)}} & \dots & \underset{\substack{1\le\rho\le P-1\\ K_u^{\rho}\in K_{i*}^{(p)}}}{\circ} a_{K_u^{(\rho)},L_n^{(Q)},H_g^{(R)}} \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
K_m^{(P)} & a_{K_m^{(P)},L_1^{(Q)},H_g^{(R)}} & \dots & \underset{\substack{1\le\sigma\le Q-1\\ L_v^{\sigma}\in L_{j*}^{(q)}}}{\circ} a_{K_m^{(P)},L_j^{(\sigma)},H_g^{(R)}} & \dots & a_{K_m^{(P)},L_n^{(Q)},H_g^{(R)}}
\end{array} \;|\; H_g^{(R)}\in H\right\},$$

where $K_i^{(P)}\subset K$ for $1\le p\le P$, $L_j^{(Q)}\subset L$ for $1\le q\le Q$.
Similar are the definitions of $(\circ)-\alpha_{(\langle K,K_i^{(P)},p\text{ -layer }\rangle,\langle H,H_g^{(R)},r\text{ -layer }\rangle)}$, $(\circ)-\alpha_{(\langle H,H_g^{(R)},r\text{ -layer }\rangle,\langle L,L_j^{(Q)},q\text{ -layer }\rangle)}$ and their generalizations.

2.2.3. *Hierarchical operators over 3D-MLEIMs.* In subsection 2.1.5, three hierarchical operators are extended over 3D-EIMs. Let us generalize these three operators over 3D-MLEIMs. 3D-MLEIM A can be represented as 3D-EIM with incorporated submatrices of multiple layers (levels). Its element $a_{K_i^{(P)},L_j^{(Q)},H_g^{(R)}}$ $= [K_i^{(P)}, L_j^{(Q)}, H_g^{(R)}, \{a_{K_{i,\omega}^{(P)},L_{j,b}^{(Q)},H_{g,c}^{(R)}}\}]$, i.e., is an IM by itself.

Then, $a_{K_{i,\omega}^{(P)},L_{j,b}^{(Q)},H_{g,c}^{(R)}} = [K_{i,\omega}^{(P)}, L_{j,b}^{(Q)}, H_{g,c}^{(R)}, \{a_{K_{i,\omega_x}^{(P-1)},L_{j,b_y}^{(Q-1)},H_{g,c_z}^{(R-1)}}\}]$ is also an index matrix and its elements are also index matrices, where $K_i^{(P)} \cap K_{i,\omega_x}^{(P-1)} = L_j^{(Q)} \cap L_{j,b_y}^{(Q-1)} = H_g^{(R)} \cap H_{g,c_z}^{(R-1)} = \emptyset$. The first hierarchical operator over 3D-MLEIM A, defined above, has the following form:

$$A|(a_{K_{i,\omega}^{(P)},L_{j,b}^{(Q)},H_{g,c}^{(R)}}) = [(K - \{K_{i,\omega}^{(P)}\}) \cup \{K_{i,\omega_1}^{(P-1)}, K_{i,\omega_2}^{(P-1)}, ..., K_{i,\omega_{I\Omega A}}^{(P-1)}\}, (L - \{L_{j,b}^{(Q)}\})$$

$$\cup\{L_{j,b_1}^{(Q-1)}, L_{j,b_2}^{(Q-1)}, ..., L_{j,b_{JB}}^{(Q-1)}\}, (H-\{H_{g_z}^{(R)}\})\cup\{H_{g,c_1}^{(R-1)}, H_{g,c_2}^{(R-1)}, ..., H_{g,c_{GC}}^{(R-1)}\}, \{c_{t_u,v_w,s_d}\}],$$

where c_{t_u,v_w,s_d}

$$= \begin{cases} a_{K_i^{(P)},L_j^{(Q)},H_g^{(R)}}, & \text{if } t_u = K_i^P \in K - \{K_{i,\omega}^P\}, \\ & v_w = L_j^Q \in L - \{L_{j,b}^Q\} \text{ and } s_d = H_g^{(R)} \in H - \{H_{g,z}^{(R)}\} \\ a_{K_{i,\omega_x}^{(P-1)},L_{j,b_y}^{(Q-1)},H_{g,c_z}^{(R-1)}}, & \text{if } t_u = K_{i,\omega_x}^{(P)} \in \{K_{i,\omega_1}^{(P-1)}, ..., K_{i,\omega_{I\Omega}}^{(P-1)}\} \\ & v_w = L_{j,b_z}^{(Q)} \in \{L_{j,b_1}^{(Q-1)}, ..., L_{j,b_{GB}}^{(Q-1)}\} \\ & \text{and } o_e = H_{g,c_z}^{(R)} \in \{H_{g,c_1}^{(R-1)}, ..., H_{g,c_{GC}}^{(R-1)}\} \\ 0, & \text{otherwise} \end{cases}.$$

The second and third operators may be applied over 3D-MLEIM $A = [K, L, H, \{a_{K_i^{(P)},L_j^{(Q)},H_g^{(R)}}\}]$ analogically.

We can extend the second hierarchical operator over 3D-MLEIM A as follows:

$$A^*|(a_{K_{i,\omega}^{(P)},L_{j,b}^{(Q)},H_{g,c}^{(R)}}; (p, q, r))$$
$$= A^*|(...A^*|(a_{K_{i,\omega}^{(P)},L_{j,b}^{(Q)},H_{g,c}^{(R)}})...),$$

where $1 \le p \le P, 1 \le q \le Q, 1 \le r \le R$. The operator will be applied consistently to levels $P, P-1, ..., p$ of dimension K, $Q, Q-1, ..., q$ of dimension Q and $R, R-1, ..., r$ of dimension R.

The third operator may be extended also as follows: $A^*|(p, q, r) = A^*|(...A^*|...)$, where $1 \le p \le P, 1 \le q \le Q, 1 \le r \le R$. This operator will be applied consistently to levels $P, P-1, ..., p$ of dimension K, $Q, Q-1, ..., q$ of dimension Q and $R, R-1, ..., r$ of dimension R. The operator $A^*|(0, 0, 0)$ will unroll the matrix A in all dimensions.

3. OLAP-cube

OLAP (On-Line Analytical Processing) technology, a term coined by Codd [9] (1993), provides interactive query-driven analysis of accumulated and consolidated business data for the purpose of decision making and knowledge extraction. On-line Analytical Processing Server (OLAP) is based on the multidimensional data model. It allows managers, and analysts to get an insight of the information through fast, consistent, and interactive access to information. These kinds of analyses can detect trends and anomalies, make projections, and make business decisions [10]. In OLAP, information is viewed conceptually as cubes that consist of descriptive categories (dimensions) and quantitative values (measures) [1, 11, 19, 12]. In the scientific literature, measures are at times called variables, metrics, properties, attributes, or indicators. Most business and scientific dimensions have a hierarchical structure. Different attributes along each dimension are often organized in hierarchical structures that determine the different levels in which data can be further analyzed [12]. A concept hierarchy of OLAP is an order relation between a set of attributes of a concept or dimension. It can be manually (users or experts) or automatically generated (statistical analysis).

For example, within the dimension “Bookshop”, one may have levels composed of BookshopName, Regional Manager and Owner (the following Fig. 1 illustrates this hierarchy)

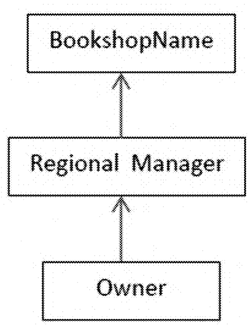

FIG. 1. View of the dimension “Bookshop” hierarchy

OLAP operators take a data cube as an input and output a new cube. These operations are defined at logical level and have to be implemented in a visual framework in the form of navigation or other interaction options [13].

3.1. **OLAP solutions.** Steps of OLAP-solutions at a high level include the following [14]:

- Understanding the current and ideal data flow;
- Defining cubes;
- Defining dimensions, members, and links;
- Defining dimension levels and/or hierarchies;
- Defining aggregations and other formulas.

For example, let us create an OLAP-cube as follows:

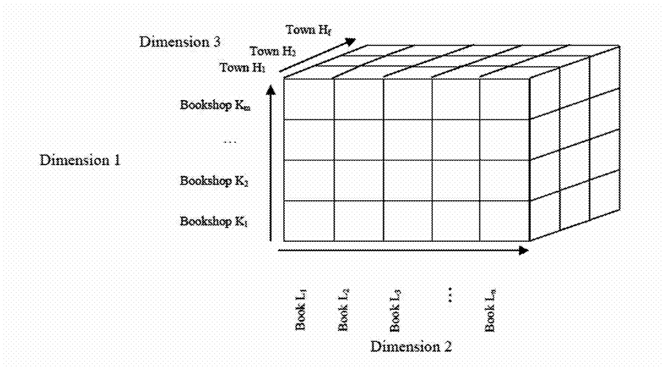

FIG. 2. Example OLAP-cube with 3 dimensions

- The cube performs sales volume as a function of books, bookshops and locations.
- Dimensions hierarchical concepts are: "Books", "Bookshops" and "Locations". The structure of the Bookshop-cube is presented on the Fig.3. The hierarchy of the dimension "Location" is *Town-Country*, the hierarchy of the dimension "Bookshops" is *Bookshop Name-Regional Manager-Owner* and the hierarchy of the dimension "Books" is *Title-Publisher-Genre*:

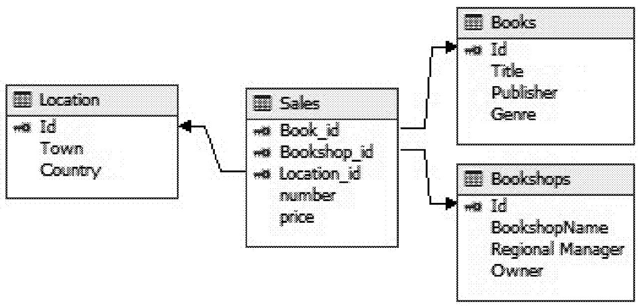

FIG. 3. Hierarchy in the dimensions

3.2. **Some basic operations over OLAP-cube via 3D-EIMs.** In terms of 3D-EIMs the above example of OLAP-cube, presented in subsection 3.1 becomes the following: let us create matrix A (3D-MLEIM with P-levels (layers) of use of a dimension K, Q-levels (layers) of use of a dimension L and R-levels (layers) of use of a dimension H, as follows:

$$A = [K, L, H, \{a_{K^{(p)}_{i,d}, L^{(q)}_{j,b}, H^{(r)}_{g,c}}\}]$$

$$= \left\{ \begin{array}{c|ccccc} H^{(R)}_g \in H & L^{(Q)}_1 & \dots & L^{(Q)}_j & \dots & L^{(Q)}_n \\ \hline K^{(P)}_1 & a_{K^{(P)}_1, L^{(Q)}_1, H^{(R)}_g} & \cdots & a_{K^{(P)}_1, L^{(Q)}_j, H^{(R)}_g} & \cdots & a_{K^{(P)}_1, L^{(Q)}_n, H^{(R)}_g} \\ \vdots & \vdots & \dots & \vdots & \dots & \vdots \\ K^{(P)}_i & a_{K^{(P)}_i, L^{(Q)}_1, H^{(R)}_g} & \cdots & a_{K^{(P)}_i, L^{(Q)}_j, H^{(R)}_g} & \cdots & a_{K^{(P)}_i, L^{(Q)}_n, H^{(R)}_g} \\ \vdots & \vdots & \dots & \vdots & \dots & \vdots \\ K^{(P)}_m & a_{K^{(P)}_m, L^{(Q)}_1, H^{(R)}_g} & \cdots & a_{K^{(P)}_m, L^{(Q)}_j, H^{(R)}_g} & \cdots & a_{K^{(P)}_m, L^{(Q)}_n, H^{(R)}_g} \end{array} \right\}$$

where

$$K = \{K^{(P)}_1, K^{(P)}_2, \dots, K^{(P)}_i, \dots, K^{(P)}_m\},$$

presents the dimension "Bookshops";

$$K^{(P)}_{i,1} \cap K^{(P)}_{i,2} = \emptyset, \text{ for } 1 \le i_1 \le m \text{ and } 1 \le i_2 \le m.$$

The level categories are *BookshopName/Regional Manager/Owner.*

$\{$ Bookshopname $\}K^{(P)}_i = \left\{K^{(P-1)}_{i,1}, K^{(P-1)}_{i,2}, \dots, K^{(P-1)}_{i,x}, \dots, K^{(P-1)}_{i,I}\right\}$ for $1 \le i \le m$

presents lower levels of the hierarchy in this dimension.

$$\dots K^{(1)}_u = \left\{K^{(0)}_{u,1}, K^{(0)}_{u,2}, \dots, K^{(0)}_{u,U}\right\}$$

i.e. p-th layer of dimension K of the multilayer matrix, where $(1 \le p \le P)$, is performed by

$$K^{(p)}_{u_*} = \left\{K^{(p-1)}_{u_*,1}, K^{(p-1)}_{u_*,2}, \dots, K^{(p-1)}_{u_*,U_*}\right\} \text{ for } 1 \le p \le P$$

$$L = \{L^{(Q)}_1, L^{(Q)}_2, \dots, L^{(Q)}_j, \dots, L^{(Q)}_n\},$$

presents the dimension "Books". The level category are *Title/Publisher/Genre/ Unit_price*;

$$L^{(Q)}_{j,1} \cap L^{(Q)}_{j,2} = \emptyset, \text{ for } 1 \le j, 1 \le n \text{ and } 1 \le j, 2 \le n.$$

$$\{ \text{ Book } \}L^{(Q)}_j = \{L^{(Q-1)}_{j,1}, L^{(Q-1)}_{j,2}, \dots, L^{(Q-1)}_{j,y}, \dots, L^{(Q-1)}_{j,J}\} \text{ for } 1 \le j \le n.$$

presents lower levels of the hierarchy in this dimension.

$$\dots L^{(1)}_v = \left\{L^{(0)}_{v,1}, L^{(0)}_{v,2}, \dots, l^{(0)}_{v,V}\right\}$$

i.e. q-th layer of dimension Q of the multilayer matrix is performed by

$$L^{(q)}_{v_*} = \left\{L^{(q-1)}_{v_*,1}, L^{(q-1)}_{v_*,2}, \dots, L^{(q-1)}_{v_*,V_*}\right\} \text{ for } 1 \le q \le Q$$

$$H = \{H^{(R)}_1, H^{(R)}_2, \dots, H^{(R)}_g, \dots, H^{(R)}_f\},$$

presents the dimension "Location". The level category are *Country/Town* and $H^{(R)}_{g,1} \cap H^{(R)}_{g,2} = \emptyset$, for $1 \le g, 1 \le f$ and $1 \le g, 2 \le f$.

$$\{ \text{ Location } \}H^{(R)}_g = \{H^{(R-1)}_{g,1}, H^{(R-1)}_{g,2}, \dots, H^{(R-1)}_{g,z}, \dots, H^{(R-1)}_{g,G}\} \text{ for } 1 \le g \le f$$

presents lower levels of the hierarchy in this dimension.

$$\dots H_w^{(1)} = \left\{ H_{w,1}^{(0)}, H_{w,2}^{(0)}, \dots, H_{w,W}^{(0)} \right\}$$

i.e. r-th layer of dimension H of the multilayer matrix is performed by

$$H_{w_*}^{(r)} = \left\{ H_{w_{*,1}}^{(r-1)}, H_{w_{*,2}^{(r-1)}}, \dots, H_{w_*,W_*}^{(r-1)} \right\} \text{ for } 1 \le r \le R$$

and ($\{K, L, H\} \subset I^*$), and for $1 \le i \le I,\ 1 \le j \le J,\ 1 \le g \le G,\ 1 \le p \le P,\ 1 \le q \le Q,\ 1 \le r \le R,\ 1 \le d \le I,\ 1 \le b \le J,\ 1 \le c \le G : a_{K_{i,d}^{(p)}, L_{j,b}^{(q)}, H_{g,c}^{(r)}} \in X$, $K_{i,0}^{(p)} \notin K$, $L_{j,0}^{(q)} \notin L$ and $H_{g,0}^{(r)} \notin H$.

3.2.1. *Operation "Roll-up".* **A)**Definition

Operation "Roll-up" performs aggregation on a data cube by climbing up the concept hierarchy or by reduction of the dimension. These techniques can be mixed [14].

The "Roll-up" (Fig. 4) operation performs selection of two or more dimensions on an OLAP cube, using a criterion, and returns a new subcube. It groups cells in the Cube based on an aggregation hierarchy.

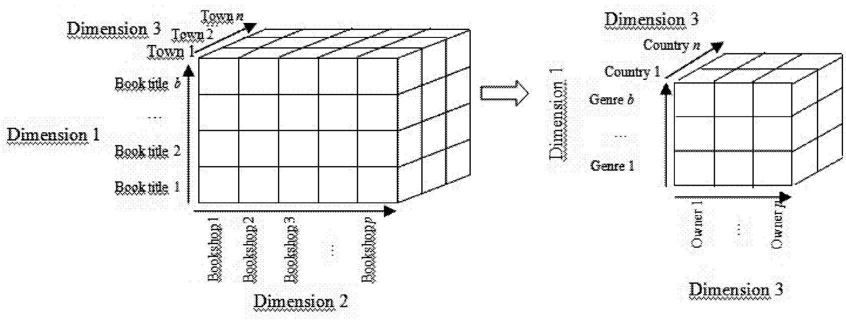

FIG. 4. Operation "Roll-up"

B)Presentation of the operation "Roll-up" by the index matrices In terms of theory of 3D-EIMs this operation for the example of Fig. 4 can be performed as follows: Let a matrix A (3D-MLEIM) be given, as defined in subsection 3.2.

The presentation of the operation "Roll-up" in terms of index matrices is realized by aggregating operators, which definition is given in section 2 of the article: $\{$ $(\circ)$–$\alpha_{(K,K_*,p \text{ -layer})}$;$(\circ)$–$\alpha_{(L,L_*,q \text{ -layer})}$; $(\circ)$–$\alpha_{(H,H_*,r \text{ -layer})}$;$(\circ)$–$\alpha_{(K,K_*,P_*)}$; $(\circ)$–$\alpha_{(L,L_*,Q_*)}$ and $(\circ) - \alpha_{(H,H_*,R_*)}\}$-aggregations.

Summarized the operation "Roll-up" can be represented, as follows:

$$\alpha_{(K,K_*,\text{ p-layer},\circ)}(A, W_*) \oplus_{(\vee,\wedge)} \alpha_{(L,L_*,\text{ q-layer},\circ)}(A, U_*)$$

$$\oplus_{(\vee,\wedge)} \alpha_{(H,H_*,\text{ r-layer},\circ)}(A, V_*)$$

where $K_* \subseteq K$ and $K_* = \{K^{(P)}_{w_1}, \ldots, K^{(P)}_{w_x}, \ldots, K^{(P)}_{w_W}\}$,
$W_* = \{K^{(p)}_{w_1,0}, \ldots, K^{(p)}_{w_x,0}, \ldots, K^{(p)}_{w_t,0}\} \notin K$ for $1 \leq p \leq P$;
$L_* \subseteq L$ and $L_* = \{L^{(Q)}_{u_1}, \ldots, L^{(Q)}_{u_y}, \ldots, L^{(P)}_{u_U}\}$,
$U_* = \{L^{(q)}_{u_1,0}, \ldots, L^{(q)}_{u_y,0}, \ldots, L^{(q)}_{u_U,0}\} \notin L$ for $1 \leq q \leq Q$;
$H_* \subseteq H$ and $H_* = \{H^{(R)}_{v_1}, \ldots, H^{(R)}_{v_z}, \ldots, H^{(R)}_{v_V}\}$,
$V_* = \{H^{(r)}_{v_1,0}, \ldots, H^{(r)}_{v_z,0}, \ldots, H^{(r)}_{v_V,0}\} \notin H$ for $1 \leq r \leq R$;

or $\alpha_{(K,K_*,P_*,\circ)}(A, W_*) \oplus_{(\vee,\wedge)} \alpha_{(L,L_*,Q_*,\circ)}(A, U_*) \oplus_{(\vee,\wedge)} \alpha_{(H,H_*,R_*,\circ)}(A, V_*)$

where $K_* \subseteq K$ and $K_* = \{K^{(P)}_{w_1}, \ldots, K^{(P)}_{w_x}, \ldots, K^{(P)}_{w_W}\}$, $W_* = \{K^{(p_1)}_{w_1,0}, \ldots, K^{(p_x)}_{w_x,0}, \ldots, K^{(p_W)}_{w_W,0}\} \notin K$ and $P_* = \{p_1, \ldots, p_x, \ldots, p_W\}$, where $p_x \in \{1 \ldots, P\}$ for $1 \leq x \leq W$;
$L_* \subseteq L$ and $L_* = \{L^{(Q)}_{u_1}, \ldots, L^{(Q)}_{u_y}, \ldots, L^{(Q)}_{u_U}\}$, $U_* = \{L^{(q_1)}_{u_1,0}, \ldots, L^{(q_y)}_{u_y,0}, \ldots, L^{(q_U)}_{u_U,0}\} \notin L$ and $Q_* = \{q_1, \ldots, q_y, \ldots, q_Q\}$, where $q_y \in \{1 \ldots, Q\}$ for $1 \leq y \leq U$;
$H_* \subseteq H$ and $H_* = \{H^{(R)}_{v_1}, \ldots, H^{(R)}_{v_z}, \ldots, H^{(R)}_{v_V}\}$, $V_* = \{H^{(r_1)}_{v_1,0}, \ldots, H^{(r_z)}_{v_y,0}, \ldots, H^{(r_V)}_{v_V,0}\} \notin H$ and $R_* = \{r_1, \ldots, r_z, \ldots, r_V\}$, where $r_z \in \{1 \ldots, P\}$ for $1 \leq z \leq V$;

C)Examples for operation "Roll-up":

- MDX query1: The first MDX query presents a standard case of "roll-up" operation. It aggregated the highest levels from the dimensions "Books", "Bookshops" and "Location".
  MDX query1:
  SELECT NON EMPTY ({[Location].[Hierarchy]},{[Books].[Hierarchy]}) ON COLUMNS, [Bookshops].[Hierarchy] ON ROWS
  FROM [Bookshops] WHERE [Measures].[Sales Count];
  Result: The query presents the number of all sold books in the cube "Bookshops" (Fig. 5). The operation is performed using three dimensions - "Books", "Bookshops" and "Location".

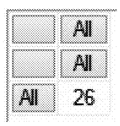

FIG. 5. Result of "roll-up" operation aggregating the highest levels from be "Bookshops" cube

- MDX query2: The next query can be used to present the meaning of the operation "roll-up" again. It returns all members of level "Country" from the hierarchy of the dimension Location according the members in highest levels by the dimensions "Bookshops" and "Books" (Fig.6 and Fig.7).
  MDX query2:
  SELECT DrillUpLevel([Books].[Hierarchy]) ON COLUMNS,
  NON EMPTY ({[Location].[Hierarchy].[Country]}) ON ROWS
  FROM [Bookshops] WHERE [Measures].[Sales Count];

Result:

	All
Bulgaria	13
Englang	4
Poland	5
Turkey	4

FIG. 6. All sold books in all countries - the dimensions "Locations" and "Books" are used

SELECT crossjoin({[Books].[Hierarchy]}, {[Bookshops].[Hierarchy]}) ON COLUMNS,
NON EMPTY ({[Location].[Hierarchy].[Country]}) ON ROWS
FROM [Bookshops]
WHERE [Measures].[Sales Count];

	All
	All
Bulgaria	13
Englang	4
Poland	5
Turkey	4

FIG. 7. All sold books in all countries - the dimensions "Locations", "Books" and "Bookshops" are used

- MDX query3: This query presents the same example with a small difference. The sold books are distributed by the members of the level "Genre" for the dimension "Books". The aim of this illustration is to presents the sold books in countries by genre. We will note that our Bookshop cube is made only for these examples. This is the purpose of the presence of the values "null" which we know from the theory of the relational databases. When the cube is used for some real application and it has many null values, this presentation can lead to sparse cube or database explosion. It is not recommended.
  MDX query3:
  SELECT NON EMPTY([Books].[Hierarchy].children)
  ON COLUMNS,
  NON EMPTY {[Location].[Hierarchy].children} ONROWS
  FROM [Bookshops] WHERE [Measures].[Sales Count];
  *Result:*The MDX query extracts the members from the levels "Country" and "Genre" of the dimensions "Locations" and "Books" and aggregates the sum of sold books in each country from each genre.

	Children Books	Computer Book	Computer Books	Cooking Books
Bulgaria	(null)	(null)	11	2
Englang	(null)	1	1	2
Poland	1	1	3	(null)
Turkey	1	(null)	3	(null)

FIG. 8. Application of "roll-up" operation over dimensions "Location" and "Books"

- MDX query4: The following MDX query extends by adding the third dimension "Bookshops".
  MDX query4:
  SELECT NON EMPTY([Books].[Hierarchy].children) ON COLUMNS,
  NON EMPTY crossjoin({[Location].[Hierarchy].children},
  {[Bookshops].[Hierarchy].children}) ON ROWS
  FROM [Bookshops]
  WHERE [Measures].[Sales Count];
  *Result:*The result of the query presents the sold books by genre, country and owner of the bookshops (Fig.9) .We will note again that the cube "Bookshops" is only for the examples and it has a null values.

		Children Books	Computer Book	Computer Books	Cooking Books
Bulgaria	Olivia Gomez	(null)	(null)	1	(null)
Bulgaria	Stamen Dimitrov	(null)	(null)	6	2
Bulgaria	Vareri Rodev	(null)	(null)	4	(null)
Englang	Stamen Dimitrov	(null)	1	1	2
Poland	Stamen Dimitrov	(null)	(null)	1	(null)
Poland	Vareri Rodev	1	1	2	(null)
Turkey	Stamen Dimitrov	1	(null)	(null)	(null)
Turkey	Vareri Rodev	(null)	(null)	3	(null)

FIG. 9. Sold books by genre, country and owner of the bookshops

- MDX query5: The last MDX query aggregates all sold books by genre "Computer Books". The aggregation is by dimension "Books". There are 18 sold computer books in all countries from all owners of bookshops.
  MDX query5:
  SELECT DrillUpLevel ({[Books].[Hierarchy].[All Books],

  [Books].[Hierarchy].[Genre].&[Computer Books]},

  [Books].[Hierarchy].[Genre]) ON COLUMNS
  FROM [Bookshops]
  WHERE [Measures].[Sales Count];
  *Result:*The MDX query extracts the members from the levels "Country" and "Genre" of the dimensions "Location" and "Books" and aggregates the sum of sold books in each country from each genre.

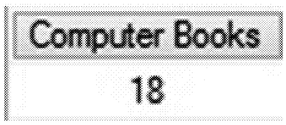

FIG. 10. The result from the query counting all the values in genre "Computer Books" is shown on Fig.10. The operation is performed using one dimensions - "Books".

3.2.2. *Operation "Drill-down".* **A)** Definition
Operation "Drill-down" is the reverse operation of "Roll-up". It is performed in either of the following ways [14]:
– Drill-down is performed by stepping down a concept hierarchy for some of the dimensions in OLAP.
– When drill-down is performed, one or more levels of the dimensions from the data cube are added.
– It navigates the data from less detailed data to highly detailed data.
The following Fig. 11 illustrates the drilled-down dimension "Books" from level "Publisher" to level "Title". The levels "Country" from dimension "Locations" and "Regional Manager" from dimension "Bookshops" are presented also.

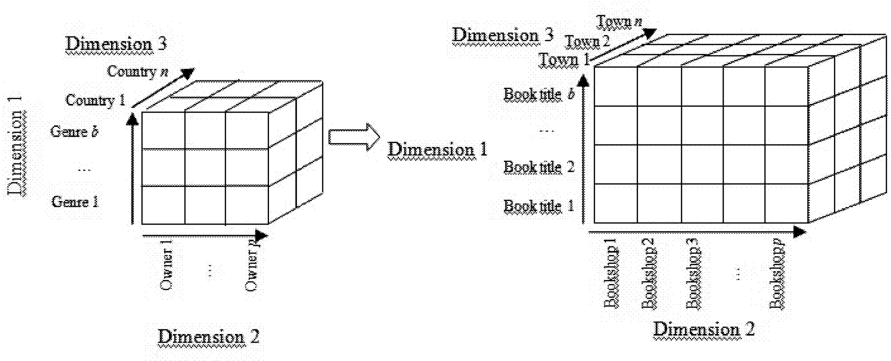

FIG. 11. Operation "Drill-down"

B)Presentation of the operation "Drill-down" by the index matrices
The operation "drill-down" over 3D-MLEIM can be expressed using hierarchical operators. In this case an element of 3D-MLEIM A is a 3D-EIM. When the operation "roll-up" is performed, the cell of A shows the total calculated value of the cells of sub-index matrix. The opposite operation "drill-down" returns the steps of aggregation. The operation "drill-through" restores the original form of the cube with the transactional data. In OLAP the steps of aggregation is frequently stored in the "tree of queries" that contains the queries that are executed on the different stages of the OLAP application. The operations "roll-up" and "drill-down" are performed depending how we navigate. If we follow the path from $a_{K^{(0)}_{i,d},L^{(q)}_{j,b},H^{(r)}_{g,c}}$ to $a_{K^{(p)}_{i,d},L^{(q)}_{j,b},H^{(r)}_{g,c}}$ to $a_{K^{(P)}_{i,d},L^{(q)}_{j,b},H^{(r)}_{g,c}}$, where $1 \leq p \leq P$, the operation roll-up executes by

the dimension K. If we follow the way back, the operation drill-down by the dimension K is implemented. Summary the OLAP cube can be represented as multilayer index matrix, which contains itself several 3D-EIM (Fig. 12).

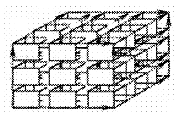

FIG. 12. Multilayer index matrix

Let us present the hierarchical operators, extended in subsection 2.2.3:
3D-MLEIM A can be represented as a 3D-EIM with incorporated matrices of multiple layers (levels). Its element $a_{K_i^{(P)},L_j^{(Q)},H_g^{(R)}}$
$= [K_i^{(P)}, L_j^{(Q)}, H_g^{(R)}, \{a_{K_{i,\omega}^{(P)},L_{j,b}^{(Q)},H_{g,c}^{(R)}}\}]$ i.e. is an IM by itself.

Then $a_{K_{i,\omega}^{(P)},L_{j,b}^{(Q)},H_{g,c}^{(R)}} = [K_{i,\omega}^{(P)}, L_{j,b}^{(Q)}, H_{g,c}^{(R)}, \{a_{K_{i,\omega_x}^{(P-1)},L_{j,b_y}^{(Q-1)},H_{g,c_z}^{(R-1)}}\}]$ is also an index matrix and its elements is also index matrices, where $K_i^{(P)} \cap K_{i,\omega_x}^{(P-1)} = L_j^{(Q)} \cap L_{j,b_y}^{(Q-1)} = H_g^{(R)} \cap H_{g,c_z}^{(R-1)} = \emptyset$.

The first hierarchical operator over 3D-MLEIM A, has the following form, presented in the subsection 2.2.3:

$$A|(a_{K_{i,\omega}^{(P)},L_{j,b}^{(Q)},H_{g,c}^{(R)}})$$
$$= [(K - \{K_{i,\omega}^{(P)}\}) \cup \{K_{i,\omega_1}^{(P-1)}, K_{i,\omega_2}^{(P-1)}, ..., K_{i,\omega_{I\Omega}}^{(P-1)}\}, (L - \{L_{j,b}^{(Q)}\})$$
$$\cup\{L_{j,b_1}^{(Q-1)}, L_{j,b_2}^{(Q-1)}, ..., L_{j,b_{JB}}^{(Q-1)}\}, (H - \{H_{g_z}^{(R)}\}) \cup \{H_{g,c_1}^{(R-1)}, H_{g,c_2}^{(R-1)}, ..., H_{g,c_{GC}}^{(R-1)}\},$$
$$\{c_{t_u,v_w,s_d}\}].$$

The second hierarchical operators may be applied over 3D-MLEIM
$A = [K, L, H, \{a_{K_i^{(P)},L_j^{(Q)},H_g^{(R)}}\}]$ and have the forms:

$$A^*|(a_{K_{i,\omega}^{(P)},L_{j,b}^{(Q)},H_{g,c}^{(R)}}) = [(K - \{K_{i,\omega}^{(P)}\}) \cup \{K_{i,\omega_1}^{(P-1)}, K_{i,\omega_2}^{(P-1)}, ..., K_{i,\omega_{I\Omega}}^{(P-1)}\}, (L - \{L_{j,b}^{(Q)}\})$$
$$\cup\{L_{j,b_1}^{(Q-1)}, L_{j,b_2}^{(Q-1)}, ..., L_{j,b_{JB}}^{(Q-1)}\}, (H - \{H_{g,z}^{(R)}\}) \cup \{H_{g,c_1}^{(R-1)}, H_{g,c_2}^{(R-1)}, ..., H_{g,c_{GC}}^{(R-1)}\},$$
$$\{c_{t_u,v_w,s_d}\}],$$
$$\text{and } A^*|(a_{K_{i,\omega}^{(P)},L_{j,b}^{(Q)},H_{g,c}^{(R)}}; (p, q, r)),$$

where $1 \le p \le P, 1 \le q \le Q, 1 \le r \le R$. The last operator will be applied consistently to levels $P, P-1, ..., p$ of dimension K, $Q, Q-1, ..., q$ of dimension Q and $R, R-1, ..., r$ of dimension R.

These two operators (more precisely, the second modified hierarchical operator if we won't to lost information) can be applied when we navigate from the path $a_{K_{i,d}^{(P)},L_{j,b}^{(q)},H_{g,c}^{(r)}}$ to $a_{K_{i,d}^{(p)},L_{j,b}^{(q)},H_{g,c}^{(r)}}$ to $a_{K_{i,d}^{(0)},L_{j,b}^{(q)},H_{g,c}^{(r)}}$, where $1 \le p \le P$. The summarized cells are calculated as result of "roll-up" operation. The third hierarchical operators have the forms as follows: A^* and $A^*|((p, q, r)) = A^*|(...A^*|...)$, where $1 \le p \le P, 1 \le q \le Q, 1 \le r \le R$. The operator $A^*|(0, 0, 0)$ will drill-down the matrix A in all dimensions.

C) *Examples for operation "Drill-down":*

- MDX query1: The first example drills down on "OReilly" and "Microsoft Press" members on the level "Publisher" to the level "Title" from dimension "Bookshops".
  MDX query1:
  SELECT {[Measures].[Sales Count]} ON COLUMNS,
  drilldownlevel ({[Books].[Hierarchy].[Publisher].&[O'Reilly],

  [Books].[Hierarchy].[Publisher].&[Microsoft Press]},

  [Books].[Hierarchy].[Publisher]) ON ROWS
  FROM [Bookshops];
  Result: The result of the query drill downs the book titles from publishers OReilly and Microsoft Press no matter in which country they were purchased and who is the owner of the bookstore (Fig.13).

	Children Books	Computer Book	Computer Books	Cooking Books
Bulgaria	(null)	(null)	11	2
Englang	(null)	1	1	2
Poland	1	1	3	(null)
Turkey	1	(null)	3	(null)

FIG. 13. The publishers "OReilly" and "Microsoft Press"are drilled down to book titles

- MDX query1a: This example drills down on OReilly and Microsoft Press members to the title level and returns the top 1 book based on the measure Sales Count:
  MDX query1a:
  SELECT {[Measures].[Sales Count]} ON COLUMNS,
  drilldownleveltop ({[Books].[Hierarchy].[Publisher].&[O'Reilly],

  [Books].[Hierarchy].[Publisher].&[Microsoft Press] }, 1,

  [Books].[Hierarchy].[Publisher],

  [Measures].[Sales Count]) ON ROWS
  FROM [Bookshops];
  Result: "Hadoop: The Definitive Guide" is the book that is sold in most units from the publisher OReilly and the book "Introducing Microsoft SQL Server 2014" is the first sold book from publisher Microsoft Press (Fig.14).

	Sales Count
O'Reilly	5
Hadoop: The Definitive Guide	2
Microsoft Press	2
Introducing Microsoft SQL server 2014	1

FIG. 14. The publishers "OReilly" and "Microsoft Press" are drilled down to the book that is sold in most units

- MDX query1b: This example drills down on "OReilly" and "Microsoft Press" members to the title level and returns the bottom one book based on the measure Sales Count:
  MDX query1b:
  SELECT [Measures].[Sales Count] ON COLUMNS,
  drilldownlevelbottom([Books].[Hierarchy].[Publisher].&[O'Reilly],

  [Books].[Hierarchy].[Publisher].&[Microsoft Press], 1,

  [Books].[Hierarchy].[Publisher], [Measures].[Sales Count]) ON ROWS

  FROM [Bookshops];
  Result: The MDX query extracts the bottom number of the sold books from publishers OReilly and Microsoft Press (Fig.15).

	Sales Count
O'Reilly	5
JavaScript: The Good Parts	1
Microsoft Press	2
Programming Microsoft SQL Server 2012	1

FIG. 15. The publishers "OReilly" and "Microsoft Press" are drilled down to the book that is sold in most units

- MDX query2: More detailed presentation of the levels in drill-down operation are presented in the next MDX query (Fig.16). The member "Bulgaria" from level "Country" is drilled down to level "Town" with members "Burgas", "Plovdiv" and "Sofia".
  MDX query2:
  WITH MEMBER [Measures].[Level] AS

  [Location].[Hierarchy].currentmember.level.ordinal
  SELECT {[Measures].[Sales Count],[Measures].[Level]} ON COLUMNS,
  nonempty (drilldownmember({[Location].[Hierarchy].[Country].[Bulgaria]},
  {[Location].[Hierarchy].[Country].[Bulgaria],

  [Location].[Hierarchy].[Town]},recursive),

  [Measures].[Sales Count]) ON ROWS
  FROM [Bookshops]

Result: The MDX query drill downs the member from level "Countries" to the members from level "Town" from dimension "Books"-(Fig.16).

	Sales Count	Level
Bulgaria	13	1
Burgas	5	2
Plovdiv	5	2
Sofia	3	2

FIG. 16. The result from the drill-down query

- MDX query3: The next query drill downs the dimension Books from level "Publisher" to level "Title" by country and owner of the bookshops:
  MDX query3:
  SELECT NON EMPTY CROSSJOIN ({[Location].[Hierarchy].[Country]}, {[Bookshops].[Hierarchy].[Regional Manager]}) ON COLUMNS,
  drilldownlevel ([Books].[Hierarchy].[Publisher].&[O'Reilly],

  [Books].[Hierarchy].[Publisher].&[Springer]}, [Books].[Hierarchy].[Publisher])
  ON ROWS
  FROM [Bookshops]
  WHERE {[Measures].[Sales Count]}
  *Result:*The result of the query presents the drilled down sold book titles from publishers "OReilly" and "Microsoft Press" by country and owner of the bookshops (Fig.17).

	Bulgaria	Bulgaria	Englang	Poland	Poland	Turkey
	Ivan Ivanov	Qna Dimova	Ivan Ivanov	Ivan Ivanov	Valeria Dimitrova	Valeria Dimitrova
O'Reilly	2	1	1	(null)	1	(null)
C# 5.0 in a Nutshell: The Definitive Reference	(null)	(null)	1	(null)	(null)	(null)
Hadoop: The Definitive Guide	1	(null)	(null)	(null)	1	(null)
JavaScript: The Definitive Guide	(null)	1	(null)	(null)	(null)	(null)
JavaScript: The Good Parts	1	(null)	(null)	(null)	(null)	(null)
Springer	1	(null)	(null)	1	(null)	1
Introduction in Computer Design	1	(null)	(null)	1	(null)	1

FIG. 17. The drilled down book titles from publishers "OReilly" and "Microsoft Press" by country and owner

- MDX query4: The next query returns the data from the fact table no matter what level of the hierarchy is ongoing:
  MDX query4:
  DRILLTHROUGH
  SELECT ([Books].[HierarchyBooks].[Genre].&[Children Books]) ON COLUMNS
  FROM [Bookshops2]
  WHERE [Location].[HierarchyLocation].[Country].&[Bulgaria]
  RETURN [Sales].[$Location.Id], [Sales].[$Bookshops.Id],

  [Sales].[$Books.Id], [Sales].[Number]
  Result: The result of the query presents the data from the fact table no matter what level of the hierarchy is ongoing (Fig.18).

[Sales].[$Location.Id]	[Sales].[$Bookshops.Id]	[Sales].[$Books.Id]	[Sales].[Number]
1	1	19	4
1	1	20	3
1	1	21	2
1	1	22	4
2	2	19	3
2	2	20	2
2	2	21	1
2	2	22	3
3	3	19	2
3	3	20	3
3	3	21	1
3	3	22	2

Fig. 18. The drilled through OLAP-cube

4. Conclusion

In the present paper we used the apparatus of the index matrices as a tool to represent OLAP operations and the existing index matrix operations to present OLAP-operation. In the future authors will present the rest OLAP-operations and their implementation. In a next research we will discuss the presented applications of index matrices in procedures for decision making with use of intercriteria analysis.

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